

---

# **A System of Three-Dimensional Complex Variables**

---

E. Dale Martin, Ames Research Center, Moffett Field, California

June 1986



National Aeronautics and  
Space Administration

**Ames Research Center**  
Moffett Field, California 94035

# A SYSTEM OF THREE-DIMENSIONAL COMPLEX VARIABLES

E. DALE MARTIN

NASA Ames Research Center, Moffett Field, CA 94035

**Abstract.** This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.

**1. Introduction.** Early in the nineteenth century, mathematicians began a search for a "three-dimensional complex number and its algebra" that would be a generalization of the ordinary "two-dimensional" complex number [1, p. 90]. In 1843, William R. Hamilton introduced quaternions (see [1]), an important four-dimensional generalization of complex numbers and variables. *Hypercomplex analysis* has developed mainly as a further generalization of quaternions and, as such, is often referred to as *Clifford analysis*. The recent papers [2], [3], [4], [5] supply many references, including early work by Fueter (e.g., [6]). These algebras that generalize quaternions are noncommutative.

S. Bergman [7] has introduced a method based on E. T. Whittaker's [8, p. 390] general integral solution to Laplace's equation that provides a certain generalization of analytic functions of one complex variable. However, the present state has been summarized as follows by E. T. Copson [9, p. 207]: "The theory of harmonic functions in two dimensions can be made to depend on the theory of analytic functions of a complex variable,  $x + iy$ . There is nothing corresponding to the theory of functions of a complex variable  $x + iy$  in three dimensions. The nearest approach is given by Whittaker's general solution ... of Laplace's equation."

The elements of the 3-D theory (a commutative algebra) to be described here are direct generalizations of corresponding elements of the classical 2-D theory. Therefore a direct comparison with 2-D is helpful for this description.

**2. Basics in Two Dimensions for Comparison.** A most important property of analytic functions of an ordinary complex variable is that from them are obtained vector functions  $g$  that are both *solenoidal* and *irrotational*. As a result, the components of  $g$  are harmonic functions.

Let  $\mathbf{R}$  denote the set of all real numbers and  $\mathbf{C}_2$  denote the set of all ordinary complex numbers. The complex variable  $z = x + iy$  in  $\mathbf{C}_2$  may be written also as  $z = (x, y) = (1, 0)x + (0, 1)y$ , which may be interpreted as a vector in  $\mathbf{R}^2$  with real components  $x, y$  and with basis vectors  $(1, 0) = 1$  and  $(0, 1) = i$ , whose *rules of multiplication* are:  $1^2 = 1$ ,  $1i = i1 = i$ ,  $i^2 = -1$ . However, the unit  $(1, 0) = 1$  as a factor is commonly omitted. If now  $g = \phi_1 + i\phi_2$ , in  $\mathbf{C}_2$ , is defined to be the vector (complex function) whose complex conjugate is an *analytic function*  $\bar{g} = f(z) = \phi_1 - i\phi_2$ , then the *conditions of analyticity* for  $\bar{g} = f(z)$  are the Cauchy-Riemann equations:  $\text{div } g = \phi_{1x} + \phi_{2y} = 0$  and  $\text{curl } g = \phi_{2x} - \phi_{1y} = 0$ .

(In two dimensions the result of the curl operation is defined as a scalar.) Therefore,  $g$  is solenoidal and irrotational (SandI).

Any 2-D S and I vector may be represented by a complex variable having the same form as  $g$ . For example, in 2-D ideal flow with velocity components  $v_1$  and  $v_2$  and with velocity potential  $\phi$  and stream function  $\psi$ , the velocity vector  $v = v_1 + iv_2$  and the vector  $g = \phi_1 + i\phi_2$  (where  $\phi_1 = \phi$  and  $\phi_2 = -\psi$ ) are called, respectively, the complex velocity and the complex potential. Both  $v$  and  $g$  are SandI vectors, and their respective complex conjugates  $\bar{v} = w(z) = v_1 - iv_2$  and  $\bar{g} = f(z) = \phi_1 - i\phi_2$  may be represented by analytic functions for which  $w = df/dz$ .

### 3. Definitions and Results in Three Dimensions.

DEFINITION 1. Let  $C_3$  denote the set of all "three-dimensional (3-D) numbers" of the form  $Z = 1x + \delta y + \epsilon z$ , in which (i)  $Z$  may be interpreted as a vector with basis vectors  $1, \delta, \epsilon$  and with components  $x, y, z$  in  $C_2$ ; and (ii) the rules of multiplication are as follows (or other equivalent forms of them):

$$\begin{aligned} 1^2 &= 1, & 1\delta &= \delta 1 = \delta, & 1\epsilon &= \epsilon 1 = \epsilon, \\ \delta^2 &= -\frac{1}{2}(1 + i\epsilon), & \epsilon^2 &= -\frac{1}{2}(1 - i\epsilon), & \delta\epsilon &= \epsilon\delta = -\frac{1}{2}(i\delta). \end{aligned}$$

The 3-D unit  $1$  as a factor may be omitted (as the factor  $1$  in 2-D is omitted), with  $Z$  written generally as

$$Z = x + \delta y + \epsilon z = Z_R + iZ_I,$$

where  $Z_R = x_R + \delta y_R + \epsilon z_R$  and  $Z_I = x_I + \delta y_I + \epsilon z_I$ , with  $x_R, y_R, z_R, x_I, y_I, z_I$  real.

DEFINITION 2. Let  $C'_3$  be a subset of  $C_3$  such that, for every element  $Z = x + \delta y + \epsilon z$  in  $C'_3$ , the components  $x, y, z$  are real.

Then, for  $Z$  in  $C_3$ ,  $Z_R$  and  $Z_I$  are in  $C'_3$ , and the basis vectors  $1, \delta$ , and  $\epsilon$  are in  $C'_3$ . If  $Z$  is an independent variable, for which values can be prescribed, then one can set  $Z_I = 0$ , so that  $x, y$  and  $z$  are real and  $Z$  is in  $C'_3$ .

The algebraic properties of these numbers in  $C_3$  are developed and discussed in papers by the author to be published. The multiplicative inverse,  $Z^{-1}$ , is of special significance. It can be found by setting  $Z^{-1} = a_1 + \delta a_2 + \epsilon a_3$ , where  $a_k \in C_2$ , and by requiring  $Z Z^{-1} = 1$ . It is found that there are certain nonzero values of  $Z$  for which  $Z^{-1}$  is not defined, with results including the following:

THEOREM 3. For  $Z = x + \delta y + \epsilon z$  in  $C'_3$ , the domain of definition of  $Z^{-1}$  includes all of the  $\mathbf{R}^3$  space of  $(x, y, z)$  except the origin and any of the six rays in the plane  $x = 0$  where  $\vartheta = \tan^{-1}(z/y) = (\ell-1)\pi/3$  for  $\ell = 1, 2, \dots, 6$ .

REMARK 4. The algebra of  $C_3$  is a linear algebra of order 3 over the field of ordinary complex numbers,  $C_2$ . Further,  $C_3$  is a commutative ring with unity, and not a field, since, for some nonzero elements  $Z$ , the inverse  $Z^{-1}$  is not defined.

Further discussion of  $Z^{-1}$  is beyond the scope of this note, but is included elsewhere.

DEFINITION 5. For every  $Z = x + \delta y + \epsilon z$  in  $C_3$ , denote as the bijugate of  $Z$  the element of  $C_3$  given by  $\bar{Z} = \frac{1}{2}x - \delta y - \epsilon z$ .

(The bijugate can be defined more generally.) The 3-D bijugate is in some ways analogous to the 2-D conjugate. The similar role in regard to analytic functions will be demonstrated here.

As an analogy to the variables  $z$  and  $g$  in  $\mathbf{C}_2$  described in the previous section, consider the two variables in  $\mathbf{C}_3$ :  $Z = x + \delta y + \epsilon z$  and  $G = \phi_1 + \delta\phi_2 + \epsilon\phi_3$ , which are also vectors in  $\mathbf{C}_2^3$ . Now let  $\bar{G}$  be defined to be the vector (3-D complex function) whose *bijugate* is an analytic function  $\bar{G} = F(Z) = \frac{1}{2}\phi_1 - \delta\phi_2 - \epsilon\phi_3$ . The concepts of function, limit, derivative, and analytic function can be extended, with some care, to the set  $\mathbf{C}_3$ . Then, in analogy to the Cauchy-Riemann conditions in two dimensions, the following necessary conditions for the differentiability, and hence analyticity, of  $F(Z)$  are found:

**THEOREM 6.** *For  $Z$  in some domain  $\mathbf{D}_3 \subseteq \mathbf{C}_3$ , and  $G$  in  $\mathbf{C}_3$  with components  $\phi_k$  in  $\mathbf{C}_2$  such that  $\bar{G} = F(Z)$ , the necessary conditions for analyticity of  $F(Z)$  are:*

$$\begin{aligned}\operatorname{div} G &= \phi_{1x} + \phi_{2y} + \phi_{3z} = 0, \\ \operatorname{curl} G &= 1(\phi_{3y} - \phi_{2z}) + \delta(\phi_{1z} - \phi_{3x}) + \epsilon(\phi_{2x} - \phi_{1y}) = 0,\end{aligned}$$

along with  $\phi_{1y} - i(\phi_{2z} + \phi_{3y}) = 0$  and  $\phi_{1z} - i(\phi_{2y} - \phi_{3z}) = 0$ .

Since all the components of the curl must vanish,  $G$  is an S and I vector in three dimensions. Further, if we write  $\phi_k = \phi_{kR} + i\phi_{kI}$  and  $G = G_R + iG_I$ , with the components  $\phi_{kR}$  of  $G_R$  and components  $\phi_{kI}$  of  $G_I$  real, then  $G_R$  and  $G_I$  are also S and I vectors (with the final two equations in Theorem 6 serving to connect the components  $\phi_{kR}$  of  $G_R$  to the components  $\phi_{kI}$  of  $G_I$ ). In Theorem 6,  $x$ ,  $y$ , and  $z$  are independent variables defined generally to be complex, but as independent variables, may be taken to be real (i.e.,  $Z \in \mathbf{C}'_3$ ).

**COROLLARY 7.** *If  $W = v_1 + \delta v_2 + \epsilon v_3$ , in  $\mathbf{C}_3$ , is defined to be the vector whose *bijugate* is the analytic function that is the derivative of  $F(Z)$ :  $\bar{W} = V(Z) = dF/dZ = \frac{1}{2}v_1 - \delta v_2 - \epsilon v_3$ , then  $W$  is also an S and I vector and*

$$\begin{aligned}v_1 &= \phi_{1x} = -(\phi_{2y} + \phi_{3z}), \\ v_2 &= \phi_{1y} = \phi_{2x} = i(\phi_{2z} + \phi_{3y}), \\ v_3 &= \phi_{1z} = \phi_{3x} = i(\phi_{2y} - \phi_{3z}), \\ \phi_{3y} &= \phi_{2z},\end{aligned}$$

**EXAMPLE 8.** For  $Z$  in  $\mathbf{C}'_3$  the product  $Z^2 = Z Z$ , with use of the rules of multiplication from Definition 1, is  $Z^2 = x^2 - \frac{1}{2}(y^2 + z^2) + \delta(2xy) + \epsilon(2xz) - i\delta(yz) - i\epsilon\frac{1}{2}(y^2 - z^2)$ . Then for  $F(Z) = Z^2$ , the results are  $\phi_{1R} = 2x^2 - (y^2 + z^2)$ ,  $\phi_{2R} = -2xy$ ,  $\phi_{3R} = -2xz$ ,  $\phi_{1I} = 0$ ,  $\phi_{2I} = yz$ ,  $\phi_{3I} = \frac{1}{2}(y^2 - z^2)$ , which are readily seen to satisfy Theorem 6. The two S and I vectors  $G_R$  and  $G_I$ , with respective Cartesian components  $\phi_{kR}$  and  $\phi_{kI}$ , are thus generated by  $F(Z) = Z^2$ .

The (harmonic) components of either  $G_R$  or  $G_I$  can be related to a 3-D velocity potential and general 3-D stream functions, and either  $G_R$  or  $G_I$  can be taken to be a "3-D complex potential," with the corresponding "3-D complex velocity" then being either  $W_R$  or  $W_I$ .

A primary result here is that this theoretical structure can be used to generate S and I vectors and harmonic functions in three dimensions, as can the Whittaker-Bergman method, but without integration here, as in ordinary analytic-function theory for two dimensions.

Details, proofs, and further results are in [10].

## REFERENCES

1. M. Kline, *Mathematics—The Loss of Certainty*, Oxford Univ. Press, New York, 1980.

2. F. Sommen, *Some Connections Between Clifford Analysis and Complex Analysis*, Complex Variables **1** (1982), 97–118.
3. J. Ryan, *Complexified Clifford Analysis*, Complex Variables **1** (1982), 119–149.
4. J. Bureš and V. Souček, *Generalized Hypercomplex Analysis and its Integral Formulas*, Complex Variables **5** (1985), 53–70.
5. L. V. Ahlfors, *Möbius Transformations in  $\mathbf{R}^n$  Expressed through  $2 \times 2$  Matrices of Clifford Numbers*, Complex Variables **5** (1986), 215–224.
6. R. Fueter, *Analytische Funktionen einer Quaternionenvariablen*, Commentarii Mathematici Helvetici **4** (1932), 9–20.
7. S. Bergman, *New Methods for Solving Boundary Value Problems*, Z. Angew. Math. Mech. **36** (1956), 182–191.
8. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge Univ. Press, Cambridge, England, 1963.
9. E. T. Copson, *Partial Differential Equations*, Cambridge Univ. Press, Cambridge, England, 1975.
10. E. D. Martin, *Some Elements of a Theory of Multidimensional Complex Variables*, NASA TM 88208, December 1985.

1. Report No. NASA TM 88318		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle  A SYSTEM OF THREE-DIMENSIONAL COMPLEX VARIABLES				5. Report Date June 1986	
				6. Performing Organization Code	
7. Author(s)  E. Dale Martin				8. Performing Organization Report No. A-86302	
9. Performing Organization Name and Address  Ames Research Center Moffett Field, CA 94035				10. Work Unit No.	
				11. Contract or Grant No.	
12. Sponsoring Agency Name and Address  National Aeronautics and Space Administration Washington, DC 20546				13. Type of Report and Period Covered Technical Memorandum	
				14. Sponsoring Agency Code 505-60-01	
15. Supplementary Notes Point of Contact: E. Dale Martin, M/S 202A-1, Ames Research Center, Moffett Field, CA 94035, (415)694-6587 or FTS 464-6587					
16. Abstract  This note reports some results of a new theory of multidimensional complex variables including, in particular, analytic functions of a three-dimensional (3-D) complex variable. Three-dimensional complex numbers are defined, including vector properties and rules of multiplication. The necessary conditions for a function of a 3-D variable to be analytic are given and shown to be analogous to the 2-D Cauchy-Riemann equations. A simple example also demonstrates the analogy between the newly defined 3-D complex velocity and 3-D complex potential and the corresponding ordinary complex velocity and complex potential in two dimensions.					
17. Key Words (Suggested by Author(s)) Three-dimensional numbers, Hypercomplex analysis, Linear algebra, Analytic functions, Harmonic functions, Potential theory, Fluid dynamics				18. Distribution Statement  Unlimited  Subject Category - 67	
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 7	
				22. Price* A02	